REAL ANALYSIS HOMEWORK 3

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1. Problem 1

Let $\{x_n\}$ be a bounded sequence of real numbers. Then, we can find an upper and lower bound M and m, respectively for this sequence. Thus, for all n, $x_n \in [m, M] := I_0$. Choose some $x_{n_0} \in I_0$.

Now, consider splitting I_0 as the intervals $[m + m/2] \cup [m/2, M]$. In at least one of these intervals, there exists an infinite number of points of $\{x_n\}$. Define this new subinterval as $I_1 \subset I_0$ and find $x_{n_1} \in I_1$.

We now proceed inductively: at the kth step, split the interval I_k in half and choose I_{k+1} as the subset containing an infinite number of points of $\{x_n\}$. This is possible since by construction, I_k has an infinite number of points of $\{x_n\}$ as well (so if neither half of I_k had an infinite number neither would I_k).

Choose $x_{n_{k+1}} \in I_{k+1}$ and note that the length of I_{k+1} is $\frac{m+M}{2^{k+1}}$. Thus, letting $k \to \infty$, we have an infinite sequence of nested intervals $I_0 \supset I_1 \supset I_2 \supset \dots$ where the length of I_k tends to 0. By the nested interval property, there exists $x^* \in \bigcap_n I_n$.

It remains to show the subsequence $\{x_{n_k}\}$ converges to the limit x^*. Thus, given $\epsilon > 0$, we can find k such that $\frac{m+M}{2^k} < \epsilon$, so that when $l \geq k$,

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$$|x_{n_l} - x^*| < \frac{m+M}{2^k} < \epsilon$$

So that $x_{n_k} \to x^*$ by definition.

2. Problem 2

Let $E \subset F$. Since $F \subset \bar{F}$, we have that $E \subset \bar{F}$. However, $\bar{F}$ is a closed set, and the closure of E is the smallest closed set containing E. Thus, $\bar{E} \subset \bar{F}$, and we are done.

3. Problem 3

Assume first that F is closed, and suppose for sake of contradiction that there exists some $x \notin F$ such that every neighborhood of x intersects F. Then, by definition, x is a limit point of F so that $x \in \bar{F} = F$, a clear contradiction. Thus, F^c must be open.

Conversely, argue by contraposition. Suppose F is not closed, so that we can find $x \in \bar{F} \backslash F$. By definition, every neighborhood of x must intersect F (since it belongs to the closure). However, since $x \notin F$, this means that there is no open neighborhood U of x for which $U \subset F^c$, implying that F^c is not open, so we are done.

4. Problem 4

Let $X = \{1,\ 2,\ 3,\ 4,\ 5\}$. For F_1, it is clear that $\{3\} \in \sigma(F)$. Also, $\varnothing$ and $X \in \sigma(F_1)$. Finally, $\{3\}^c = \{1,\ 2,\ 4,\ 5\} \in \sigma(F_1)$. Checking intersections and unions, we see that this exhausts the elements of $\sigma(F_1)$ so that

$$\sigma(F_1) = \{\varnothing,\ X,\ \{3\},\ \{1,\ 2,\ 4,\ 5\}\}$$

Similarly for the set $F_2 = \{\{3\},\ \{4\}\}$, we have that $\varnothing,\ X \in \sigma(F_2)$. Also, as $F_1 \subset F_2$, $\sigma(F_1) \subset \sigma(F_2)$.

By definition, $\{3\}$ and $\{4\} \in \sigma(F_2)$, which then forces their union $\{3,\ 4\} \in \sigma(F_2)$. Consequently, $\{3,\ 4\}^c = \{1,\ 2,\ 5\} \in \sigma(F_2)$, and similarly $\{4\}^c = \{1,\ 2,\ 3,\ 5\} \in \sigma(F_2)$. Checking intersections and unions we see that this exhausts all elements of $\sigma(F_2)$ (keeping in mind we've used that $\sigma(F_1) \subset \sigma(F_2)$). Therefore,

$$\sigma(F_2) = \{\varnothing,\ X, \{3\},\ \{4\},\ \{1,\ 2,\ 4,\ 5\},\ \{3,\ 4\},\ \{1,\ 2,\ 5\},\ \{1,\ 2,\ 3,\ 5\}\}$$

5. Problem 5

We first show that the sequence of $F_n(t)$ is an ascending chain. Let $x \in F_{n-1}(t)$. Then, by definition, $f_{n-1}(x) \geq t$. But $f_n(x) \geq f_{n-1}(x) \geq t$, so that $F_n(t) \supset F_{n-1}(t)$ for all t.

Using this, by definition $\lim\limits_{n\to\infty} F_n(t) := \bigcup\limits_{n=1}^{\infty} F_n(t)$.

Suppose now that $x \in \bigcup_{n=1}^{\infty} F_n(t)$. Then, for some N, $f_n(x) \geq t$ for all $n \geq N$, and since this is an increasing sequence, $f(x) \geq f_n(x) \geq t$ for all $n \in \mathbb{N}$. By definition, this shows that $x \in F(t)$, so that

$$\lim_{n\to\infty} F_n(t) \subseteq F(t)$$

To prove this inclusion is strict, consider the sequence of functions defined by $f_n(x) = -\frac{x^2}{n}$. It is clear that $f_1(x) \leq f_2(x) \leq \dots$, and furthermore we see that the f_n converge pointwise to $f \equiv 0$.

Now set $t = 0$. Then for all $n \in \mathbb{N}$, $F_n(0)$ is the set of all x such that $x^2/n \leq 0$, which is clearly just the singleton $\{0\}$. Thus, we see that $\lim\limits_{n\to\infty} F_n(0) = \{0\}$.

Now for $F(0)$, since $f(x) = 0$ for all x, $F(0)$ is in fact $\mathbb{R}$, so we clearly have strict inclusion. [1]

[1] Actually, an even simpler example shows how strict this inclusion really is: Just define $f_n(x) = -1/n$. Then $f_n \to 0$ uniformly, and $F_n(0) = \varnothing$ for all n but $F(t) = \mathbb{R}$.